Sea $f : \mathbb{C} \to \mathbb{C}$ dada por $f(z) = \dfrac{z}{1-i}$. Definimos $f^2 = f \circ f$, $f^3 = f \circ f^2$, $\ldots$, $f^{(n+1)} = f \circ f^{(n)}$ y llamamos $f^{(n)}(z) = w_n$.

(a) Si $z = i$, hallar el menor $n \in \mathbb{N}$ posible para que $w_1 + w_2 + \cdots + w_n$ sea real y calcular el valor correspondiente.

(b) Hallar $z^{1/6}$ sabiendo que $w_{200} = i$.

(a) $z = i$ $\quad \sum_{k=1}^{n} w_k \in \mathbb{R}$ ?

$$w_1 = f(z) = \frac{z}{1-i}$$

$$w_2 = f^2(z) = (f \circ f)(z) = f(f(z)) = f\left(\frac{z}{1-i}\right) = \frac{z}{(1-i)^2}$$

$$w_3 = f^3(z) = (f \circ f^2)(z) = f(f^2(z)) = f\left(\frac{z}{(1-i)^2}\right) = \frac{z}{(1-i)^3}$$

...

$$w_k = f^{(k)}(z) = \frac{z}{(1-i)^k}$$

$$\sum_{k=1}^{n} w_k = \sum_{k=1}^{n} \frac{z}{(1-i)^k} = z \underbrace{\sum_{k=1}^{n} \frac{1}{(1-i)^k}}_{S} \overset{(*)}{=}$$

(*) $$S = \frac{1}{(1-i)} + \frac{1}{(1-i)^2} + \cdots + \frac{1}{(1-i)^n}$$

$$-\frac{1}{(1-i)} S = \quad - \frac{1}{(1-i)^2} - \cdots - \frac{1}{(1-i)^n} - \frac{1}{(1-i)^{n+1}}$$

$$\left[1 - \frac{1}{(1-i)}\right] S = \frac{1}{1-i} - \frac{1}{(1-i)^{n+1}}$$

$$\frac{1-i-1}{1-i}\, S' = \frac{1}{(1-i)}\left[1-\frac{1}{(1-i)^n}\right]$$

$$S' = \frac{1}{-i}\left[1-\frac{1}{(1-i)^n}\right]$$

$$S' = i\left[1-\frac{1}{(1-i)^n}\right]$$

$$= z\cdot i\left[1-\frac{1}{(1-i)^n}\right]$$

$$\Rightarrow \underline{z=i} \Rightarrow \sum_{k=1}^{n} w_k = -\left[1-\frac{1}{(1-i)^n}\right] =$$

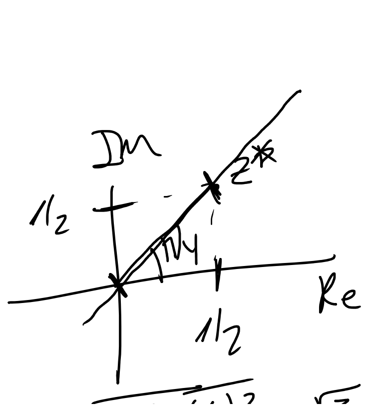
$$= \frac{1}{(1-i)^n} - 1 = \frac{(1+i)^n}{(1-i)^n(1+i)^n} - 1 =$$

$$= \frac{(1+i)^n}{[1^2-i^2]^n} - 1 = \left(\frac{1+i}{2}\right)^n - 1 \in \mathbb{R}$$

sii $\left(\frac{1+i}{2}\right)^n \in \mathbb{R}$

(*)

$$z^* = \frac{1+i}{2} = \frac{1}{2}+\frac{i}{2}$$



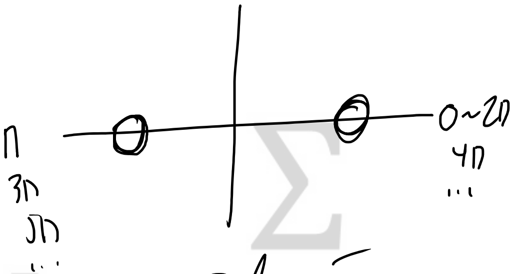
$z^* = \frac{1+i}{2} = \frac{1}{2} + \frac{i}{2}$ [illegible]

$\frac{1}{2}$

$|z^*| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$

sii $\left(\frac{\sqrt{2}}{2}{}_{\pi/4}\right)^n \in \mathbb{R}$

sii $\left(\frac{\sqrt{2}}{2}\right)^n_{\frac{n\cdot\pi}{4}} \in \mathbb{R}$




sii $\frac{n\cdot\pi}{4} = \pi \longrightarrow n = 4$ ← Solución

¿ $\sum_{k=1}^{4} w_k \in \mathbb{R}$ ?

$\left(\frac{\sqrt{2}}{2}\right)^4_{\pi} - 1 = \left(\frac{4}{16}\right)_{\pi} - 1 = -\frac{1}{4} - 1 = \frac{-5}{4}$ !!

(b) $w_{200} = i$ ¿ $z^{1/6}$ ?

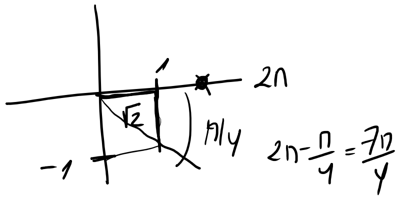
$w_{200} = \frac{z}{(1-i)^{200}} = i$

$z = i(1-i)^{200}$

$z = i\cdot(1-i)^{200} \overset{(*)}{=} i\cdot\left(\sqrt{2}_{7\pi/4}\right)^{200} = i\cdot\left(\sqrt{2}\right)^{200}_{\frac{1400\pi}{4}} =$

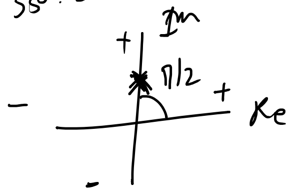
(*) [illegible] $\sqrt{2}$ [illegible]

(*) $z^* = 1 - i = \sqrt{2}_{\frac{7\pi}{4}}$




$2^{100}\left(\cos 350\pi + i \,\text{sen}\, 350\pi\right)$

$350 : 2 = 175$




$= i \cdot 2^{100}_{350\pi} = 2^{100} \cdot i$

$$z^{1/6} = \left(2^{100} \cdot i\right)^{1/6} = \left(2^{100}_{\pi/2}\right)^{1/6} = \left(2^{100/6}\right)_{\frac{\pi/2 + 2\pi k}{6}} \quad k = 0, 1, \dots, 5$$

Solución: $$z^{1/6} = \left(2^{50/3}\right)_{\frac{\pi}{12} + \frac{\pi k}{3}}, \quad k = 0, 1, 2, \dots, 5$$