3. Calcule el límite en el infinito de la sucesión A_n, siendo A_n el siguiente determinante:

$$\begin{vmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 & \cdots & 0 \\ x & 1 & -\frac{1}{3} & 0 & 0 & \cdots & 0 \\ x^2 & 0 & 1 & -\frac{1}{4} & 0 & \cdots & 0 \\ x^3 & 0 & 0 & 1 & -\frac{1}{5} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x^{n-2} & 0 & 0 & 0 & 0 & \cdots & -\frac{1}{n} \\ x^{n-1} & 0 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}.$$

$A_1 = 1$

$A_2 = \begin{vmatrix} 1 & -1/2 \\ x & 1 \end{vmatrix} = 1 + \frac{x}{2}$

$A_3 = \begin{vmatrix} 1 & -1/2 & 0 \\ x & 1 & -1/3 \\ x^2 & 0 & 1 \end{vmatrix} = 1 \cdot A_2 + x^2 \begin{vmatrix} -1/2 & 0 \\ 1 & -1/3 \end{vmatrix} = 1 \cdot \left(1 + \frac{x}{2}\right) + x^2 \cdot \left(\frac{-1}{2} \cdot \frac{-1}{3}\right) =$

$= 1 + \frac{x}{2} + \frac{x^2}{6}$

$A_4 = \begin{vmatrix} 1 & -1/2 & 0 & 0 \\ x & 1 & -1/3 & 0 \\ x^2 & 0 & 1 & -1/4 \\ x^3 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot A_3 + x^3 \cdot (-1) \begin{vmatrix} -1/2 & 0 & 0 \\ 1 & -1/3 & 0 \\ 0 & 1 & -1/4 \end{vmatrix} = 1 + \underbrace{\frac{x}{2}}_{2!} + \underbrace{\frac{x^2}{6}}_{3!} + \frac{x^3}{4!}$

...

$A_n = \begin{vmatrix} 1 & -1/2 & 0 & 0 & \cdots & 0 \\ x & 1 & -1/3 & 0 & \cdots & 0 \\ x^2 & 0 & 1 & -1/4 & \cdots & 0 \\ \vdots & & & & & \vdots \\ x^{n-1} & 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = 1 \cdot A_{n-1} (-1)^{2n} + x^{n-1} \begin{vmatrix} -1/2 & 0 & 0 & \cdots & 0 \\ 1 & -1/3 & 0 & \cdots & 0 \\ 0 & 1 & -1/4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & & -1/n \end{vmatrix} (-1)^{n+1} =$

$= A_{n-1} + x^{n-1} \cdot \frac{-1}{n} \cdot \frac{-1}{n-1} \cdot \ldots \cdot \frac{-1}{4} \cdot \frac{-1}{3} \cdot \frac{-1}{2} (-1)^{n+1} =$

$= A_{n-1} + x^{n-1} \cdot \frac{(-1)^{n-1} \cdot (-1)^{n+1}}{n!} =$

$= A_{n-1} + \frac{x^{n-1}}{n!} \cancel{(-1)^{2n}} = \frac{x^{n-1}}{n!} + A_{n-1}$

$$\boxed{A_n = \frac{x^{n-1}}{n!} + A_{n-1}}$$

... $\boxed{A_n = \frac{x^{n-1}}{n!} + \frac{x^{n-2}}{(n-1)!} + \ldots + \frac{x^3}{4!} + \frac{x^2}{3!} + \frac{x}{2!} + 1}$

$A_1 = 1$

$A_2 = 1 + \frac{x}{2}$

$A_3 = 1 + \frac{x}{2!} + \frac{x^2}{3!}$ $\Longrightarrow A_1, A_2, A_3, \ldots, A_n, \ldots$

$$\lim_{n \to \infty} A_n = \begin{cases} \lim_{n \to \infty} 1, & \text{si } x = 0 \\ (*) \lim \frac{x^{n-1}}{n!} + \ldots + \frac{x^2}{3!} + \frac{x}{2!} + 1 & \text{si } x \neq 0 \end{cases}$$

$$\lim_{n\to\infty} A_n = \begin{cases} \circledast \lim_{n\to\infty} \dfrac{x^{n-1}}{n!} + \dots + \dfrac{x^2}{3!} + \dfrac{x}{2!} + 1 & \text{si } x \neq 0 \end{cases}$$

$$\circledast \lim_{n\to\infty} \frac{x^{n-1}}{n!} + \dots + \frac{x^2}{3!} + \frac{x}{2!} + 1 = \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} =$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$= \sum_{k=1}^{\infty} \frac{x^k}{k! \cdot x} =$$

$$= \frac{1}{x} \cdot \left[\sum_{k=0}^{\infty} \frac{x^k}{k!} - 1 \right] = \frac{1}{x}\left[e^x - 1\right]$$

$$\boxed{\lim_{n\to\infty} A_n = \begin{cases} 1 & \text{si } x = 0 \\ \dfrac{e^x - 1}{x} & \text{si } x \neq 0 \end{cases}}$$