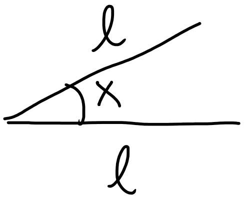
Los dos lados de un triángulo isósceles tienen una longitud ℓ, cada uno, y el ángulo x ellos es el valor de una variable aleatoria X con función de densidad proporcional a $x(\pi - x)$ en cada punto $x \in (0, \frac{\pi}{2})$. Calcular la función de densidad del área del triángulo y su esperanza.



$X \sim$

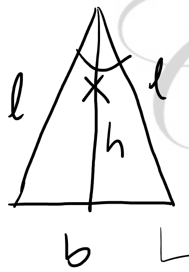
$f(x) = kx(\pi - x) \quad x \in \left(0, \frac{\pi}{2}\right)$

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1 \longrightarrow \int_0^{\pi/2} kx(\pi - x)\,dx = 1$$

$$k\int_0^{\pi/2} (x\pi - x^2)\,dx = k\left[\frac{x^2\pi}{2} - \frac{x^3}{3}\right]_0^{\pi/2} =$$

$$= k\left[\frac{\pi^3}{8} - \frac{\pi^3}{24}\right] = k \cdot \frac{2\pi^3}{24} = \frac{k\pi^3}{12} = 1$$

$$k = \frac{12}{\pi^3} \Longrightarrow \boxed{f(x) = \frac{12}{\pi^3}\, x(\pi - x), \quad x \in \left(0, \frac{\pi}{2}\right)}$$



$$A = \frac{b \cdot h}{2} = \frac{2\ell \operatorname{sen}\frac{x}{2} \cdot \ell \cos\frac{x}{2}}{2}$$

$$\cos\frac{x}{2} = \frac{h}{\ell} \longrightarrow h = \ell \cos\frac{x}{2}$$

$$\operatorname{sen}\frac{x}{2} = \frac{b/2}{\ell} \longrightarrow b = 2\ell \operatorname{sen}\frac{x}{2}$$

$$A = \frac{2\ell^2 \overbrace{\operatorname{sen}\frac{x}{2}\cos\frac{x}{2}}}{2} = \frac{\ell^2 \operatorname{sen} x}{2}$$

$\hookrightarrow \operatorname{sen} 2x = 2 \operatorname{sen} x \cos x$

$$Y = \frac{\ell^2 \operatorname{sen} X}{}$$

$$Y = \frac{\ell^2 \operatorname{sen} X}{2}$$

$$G(y) = P(Y \leq y) = P\left(\frac{\ell^2 \operatorname{sen} X}{2} \leq y\right) =$$

$$= P\left(\operatorname{sen} X \leq \frac{2y}{\ell^2}\right) = P\left(X \leq \operatorname{arcsen}\left(\frac{2y}{\ell^2}\right)\right) =$$

$$= F\left(\operatorname{arcsen}\left(\frac{2y}{\ell^2}\right)\right)$$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_0^x \frac{12}{\pi^3} \overbrace{x(\pi - x)}^{x\pi - x^2}\,dx =$$

$$= \frac{12}{\pi^3}\left[\frac{x^2\pi}{2} - \frac{x^3}{3}\right]_0^x = \frac{12}{\pi^3}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]$$

$$x \in \left(0, \frac{\pi}{2}\right)$$

$$G(y) = F\left(\operatorname{arcsen}\left(\frac{2y}{\ell^2}\right)\right) =$$

$$= \frac{12}{\pi^3}\left[\frac{\pi}{2}\operatorname{arcsen}^2\left(\frac{2y}{\ell^2}\right) - \frac{1}{3}\operatorname{arcsen}^3\left(\frac{2y}{\ell^2}\right)\right]$$

$$y \in \left(0, \frac{\ell^2}{2}\right)$$

$$g(y) = G'(y) = \frac{12}{\pi^3}\left[\frac{\pi}{\cancel{2}}\cancel{2}\operatorname{arcsen}\left(\frac{2y}{\ell^2}\right) \cdot \frac{2/\ell^2}{\sqrt{1 - \left(\frac{2y}{\ell^2}\right)^2}} - \right.$$

$$g(y) = F'(Y) = \frac{12}{\pi^3}\left[\frac{\pi}{\cancel{2}}\cancel{2}\operatorname{arcsen}\left(\frac{2y}{\ell^2}\right)\cdot\frac{}{\sqrt{1-\left(\frac{2y}{\ell^2}\right)^2}}\right.$$

$$\left. -\frac{1}{\cancel{3}}\cancel{3}\operatorname{arcsen}^2\left(\frac{2y}{\ell^2}\right)\cdot\frac{2/\ell^2}{\sqrt{1-\left(\frac{2y}{\ell^2}\right)^2}}\right] =$$

$$= \frac{12}{\pi^3}\operatorname{arcsen}\left(\frac{2y}{\ell^2}\right)\cdot\frac{2}{\sqrt{\ell^4-4y^2}}\left[\pi-\operatorname{arcsen}\left(\frac{2y}{\ell^2}\right)\right]$$

$$y \in \left(0, \frac{\ell^2}{2}\right)$$

$$E(Y) = \int_{-\infty}^{+\infty} y\cdot g(y)\,dy =$$

$$= \int_0^{\ell^2/2} y\cdot\frac{12}{\pi^2}\cdot\operatorname{arcsen}\left(\frac{2y}{\ell^2}\right)\cdot\overbrace{\frac{2/\ell^2}{\sqrt{1-\left(\frac{2y}{\ell^2}\right)^2}}\,dy}^{dt}$$

$$-\int_0^{\ell^2/2} y\,\frac{12}{\pi^3}\operatorname{arcsen}^2\left(\frac{2y}{\ell^2}\right)\overbrace{\frac{2/\ell^2}{\sqrt{1-\left(\frac{2y}{\ell^2}\right)^2}}\,dy}^{dt} =$$

$$= \left[\, t = \frac{2y}{\ell^2} \quad \begin{matrix} y=0 \to t=0 \\ y=\ell^2/2 \to t=1 \end{matrix} \quad \Rightarrow y = \frac{t\ell^2}{2} \,\right] =$$

$$dt = \frac{2}{\ell^2} dy \qquad dy = \frac{\ell^2}{2} dt$$

$$= \int_0^1 \frac{t\ell^2}{2} \frac{12}{\pi^2} \arcsin(t) \frac{1}{\sqrt{1-t^2}} dt -$$

$$- \int_0^1 \frac{t\ell^2}{2} \frac{12}{\pi^3} \arcsin^2(t) \cdot \frac{1}{\sqrt{1-t^2}} dt =$$

$$= \frac{6\ell^2}{\pi^2} \int_0^1 \frac{t}{\sqrt{1-t^2}} \arcsin(t)\, dt -$$

$$- \frac{6\ell^2}{\pi^3} \int_0^1 \frac{t}{\sqrt{1-t^2}} \arcsin^2(t)\, dt \overset{(*)}{=}$$

$(*)$

$(*)_1$ $\displaystyle\int \frac{t}{\sqrt{1-t^2}} \arcsin(t)\, dt =$

$$= \left[u = \arcsin t \qquad du = \frac{1}{\sqrt{1-t^2}} dt \right] =$$

$$= \left[\begin{array}{ll} u = \arcsin t & du = \dfrac{1}{\sqrt{1-t^2}}\,du \\ dv = t\underbrace{(1-t^2)^{-1/2}}\,dt & v = \dfrac{-1}{2}\dfrac{(1-t^2)^{+1/2}}{1/2} = \\ & = -\sqrt{1-t^2} \end{array}\right] =$$

$$= -\sqrt{1-t^2}\arcsin t - \int -1\,dt =$$

$$\boxed{= -\sqrt{1-t^2}\arcsin t + t + C}$$

$$\circledast_2 \quad \int \frac{t}{\sqrt{1-t^2}}\arcsin^2(t)\,dt =$$

$$= \left[\begin{array}{ll} u = \arcsin^2 t & du = 2\arcsin t \cdot \dfrac{1\,dt}{\sqrt{1-t^2}} \\ dv = \dfrac{t}{\sqrt{1-t^2}}\,dt & v = -\sqrt{1-t^2} \end{array}\right] =$$

$$= -\sqrt{1-t^2}\arcsin^2 t + 2\int \arcsin t\,dt =$$

$$= \left[\begin{array}{ll} u = \arcsin t & du = \dfrac{1}{\sqrt{1-t^2}}\,dt \\ dv = 1\cdot dt & v = t \end{array}\right] =$$

$$dv = 1 \cdot dt \qquad v = t$$

$$= -\sqrt{1-t^2}\,\arcsin^2 t + 2t \arcsin t - 2\underbrace{\int \frac{t}{\sqrt{1-t^2}}\,dt}$$

$$= -\sqrt{1-t^2}\,\arcsin^2 t + 2t \arcsin t + 2\sqrt{1-t^2} + C$$

$$= \frac{6\ell^2}{\pi^2}\left[1-0\right] - \frac{6\ell^2}{\pi^3}\left[\cancel{2}\cdot\frac{\pi}{\cancel{2}} - 2\right] = \frac{12\ell^2}{\pi^3}$$