PROBLEMA 4:

De una urna con r bolas negras y $N-r$ bolas blancas se extraen n bolas consecutivamente y sin reemplazamiento ($n \le N$). Calcule la esperanza y la varianza de la variable aleatoria X que da el número de bolas negras extraídas.

r Negras

$N-r$ Blancas

n extracciones

$X \equiv$ "# negras en n extracciones".

$$P(X=x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

$$E(X) = \frac{n \cdot r}{N}$$

$$V(X) = \frac{n \cdot r \cdot (N-r)(N-n)}{N^2(N-1)}$$

$$E(X) = \sum_{x \in sop^*} x \cdot P(X=x) = \sum_{x \in sop} x \cdot \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}} =$$

$^*\left[\max(0, n-(N-r)) \le x \le \min(r, n)\right]$

$$= \frac{1}{\binom{N}{n}} \sum_{x \in sop} \cancel{x} \frac{r(r-1)\cdots(r-x+1)}{\cancel{x}(x-1)\cdots 2 \cdot 1} \cdot \binom{N-r}{n-x} =$$

$$= \frac{r}{\binom{N}{n}} \boxed{\sum_{x \in sop} \binom{r-1}{x-1}\binom{N-r}{n-x}} \overset{*}{=} \frac{r}{\binom{N}{n}} \binom{N-1}{n-1} =$$

$$^*\left[\sum_{x \in sop} \frac{\binom{r-1}{x-1}\binom{N-r}{n-x}}{\binom{N-1}{n-1}} = 1\right]$$

$$\Big[\quad \binom{n-1}{} \quad \Big]$$

$$= \frac{r}{\dfrac{N!}{n!\,\cancel{(N-n)!}}} \cdot \frac{(N-1)!}{(n-1)!\,\cancel{(N-n)!}} = \frac{r}{\frac{N}{n}} = \frac{r \cdot n}{N}$$

$$E(X^2) = \sum_{x \in sop} x^2 \cdot P(X=x) = \sum_{x \in sop} x^2 \, \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}} =$$

$$= \frac{1}{\binom{N}{n}} \sum_{x \in sop} x^{\cancel{2}} \, \frac{r}{\cancel{x}} \cdot \binom{r-1}{x-1}\binom{N-r}{n-x} =$$

$$= \frac{r}{\binom{N}{n}} \sum_{x \in sop} x \binom{r-1}{x-1}\binom{N-r}{n-x} =$$

$$= \frac{r}{\binom{N}{n}} \sum_{x \in sop} \underbrace{(x-1+1)} \binom{r-1}{x-1}\binom{N-r}{n-x} =$$

$$= \frac{r}{\binom{N}{n}} \left[\sum_{x \in sop} (x-1)\binom{r-1}{x-1}\binom{N-r}{n-x} + \sum_{x \in sop} \binom{r-1}{x-1}\binom{N-r}{n-x} \right]$$

$$= \frac{r}{\binom{N}{n}} \left[\sum_{x \in sop} \cancel{(x-1)} \frac{r-1}{\cancel{x-1}} \binom{r-2}{x-2}\binom{N-r}{n-x} + \binom{N-1}{n-1} \right]$$

$$\Big[(r-1) \binom{N-2}{} + \binom{N-1}{} \Big]$$

$$= \frac{r}{\binom{N}{n}}\left[(r-1)\binom{N-2}{n-2} + \binom{N-1}{n-1}\right]$$

$$= \frac{r(r-1)}{\frac{N!}{n!\,\cancel{(N-n)!}}} \cdot \frac{(N-2)!}{(n-2)!\,\cancel{(N-n)!}} + \frac{r}{\frac{N!}{n!\,\cancel{(N-n)!}}} \cdot \frac{(N-1)!}{(n-1)!\,\cancel{(N-n)!}}$$

$$= \frac{r(r-1)\cdot n(n-1)}{N(N-1)} + \frac{r\cdot n}{N}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{r(r-1)\cdot n(n-1)}{N(N-1)} + \frac{rn}{N} - \frac{r^2n^2}{N^2}$$

$$= \frac{N\cdot r(r-1)\,n(n-1)}{N^2(N-1)} + \frac{rnN(N-1)}{N^2(N-1)} - \frac{r^2n^2(N-1)}{N^2(N-1)}$$

$$= \frac{r\cdot n\left[N(\cancel{rn} - r - n + \cancel{1}) + N^2 - \cancel{N} - \cancel{rnN} + rn\right]}{N^2(N-1)} =$$

$$= \frac{r\cdot n\left[-rN - nN + N^2 + rn\right]}{N^2(N-1)}$$

$$= \frac{r\cdot n\left[N(N-r) + n(r-N)\right]}{N^2(N-1)} =$$

$$= \frac{r\cdot n(N-r)(N-n)}{N^2(N-1)}$$