Ejercicio 02:
Sea $f : \mathbb{R}^3 \to \mathbb{R}^3$ la aplicación lineal dada por

$$f(x, y, z) = (x + y, z, -z).$$

1. Calcule una base y unas ecuaciones de la imagen y del núcleo de f.
2. Calcule la descomposición canónica de f.

$$\operatorname{Im} f \cong U/\operatorname{Ker} f$$

$$\begin{array}{ccc} U & \xrightarrow{\ f\ } & V \\ \pi \downarrow & & \uparrow i \\ U/\operatorname{Ker} f & \xrightarrow{\ g\ } & \operatorname{Im} f \end{array}$$

$$\boxed{f = i \circ g \circ \pi}$$

$$M_f \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\boxed{\begin{array}{c}\operatorname{Ker} f \\ \operatorname{Im} f\end{array}}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 0 & 1 & y \\ 0 & 0 & -1 & z \end{array}\right) \xrightarrow{F_3+F_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & \boxed{z+y} \end{array}\right)$$

$$\operatorname{Ker} f \equiv \begin{cases} x+y=0 \\ z=0 \end{cases} \equiv \begin{cases} x=-y \\ y=y \\ z=0 \end{cases}, \ y \in \mathbb{R}$$

$$\operatorname{Im} f \equiv y+z=0 \equiv \begin{cases} x=x \\ y=-z \\ z=z \end{cases} \quad x, z \in \mathbb{R}$$

$$B_{\operatorname{Ker} f} = \{(-1,1,0)\}$$

$$\{(1,0,0)(0,-1,1)\}$$

$\beta_{Im f} = \{(1,0,0)\,(0,-1,1)\}$

$\{(1,0,0),\,(0,0,1)\}$

$\longrightarrow \beta_{\mathbb{R}^3/\ker f} = \{(1,0,0)+\ker f,\,(0,0,1)+\ker f\}$

$\pi: \mathbb{R}^3 \longrightarrow \mathbb{R}^3/\ker f$

$x \rightsquigarrow [x] = x + \ker f$

$\pi(x) = x + \ker f$

$\pi(1,0,0) = [(1,0,0)] = \alpha[(1,0,0)] + \beta[(0,0,1)] = (1,0)_{\beta_{\mathbb{R}^3/\ker f}}$

$\pi(0,1,0) = [(0,1,0)] = \alpha[(1,0,0)] + \beta[(0,0,1)] = (1,0)_{\beta_{\mathbb{R}^3/\ker f}}$

$\pi(0,0,1) = [(0,0,1)] = \alpha[(1,0,0)] + \beta[(0,0,1)] = (0,1)_{\beta_{\mathbb{R}^3/\ker f}}$

$[(0,1,0)] = [(\alpha,0,\beta)]$

$[(-\alpha,1,-\beta)] = 0$

$-\alpha+1=0 \rightarrow \alpha = 1$

$-\beta=0 \rightarrow \beta = 0$

$M_\pi = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$M_n = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$g: \mathbb{R}^3/\ker f \longrightarrow \operatorname{Im} f$$

$$\{[(1,0,0)], [(0,0,1)]\} \qquad \{(1,0,0)(0,-1,1)\}$$

$$g([(1,0,0)]) = f(1,0,0) = (1,0,0) = \alpha(1,0,0) + \beta(0,-1,1) = (1,0)_{\beta \operatorname{Im} f}$$

$$g([(0,0,1)]) = f(0,0,1) = (0,1,-1) = \alpha(1,0,0) + \beta(0,-1,1) = (0,-1)_{\beta \operatorname{Im} f}$$

$$M_g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$i: \operatorname{Im} f \longrightarrow \mathbb{R}^3$$
$$x \mapsto x$$

$$i(x) = x$$

$$i(1,0,0) = (1,0,0)_e$$
$$i(0,-1,1) = (0,-1,1)_e$$

$$M_i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M_i \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$

$$g = i \circ g \circ \Pi$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & +1 \\ 0 & 0 & -1 \end{pmatrix}$$