**2.** Sean $a$ y $b$ dos números reales positivos.

a) Demostrar que si $a < b < e$ entones $a^b < b^a$.

b) Demostrar que si $e < a < b$ entonces $a^b > b^a$.

$a, b \in \mathbb{R}^{++}$ $\quad \mathbb{R}^{++} = (0, +\infty)$

a) $a^b < b^a \overset{*}{\iff} \ln a^b < \ln b^a \iff b \ln a < a \ln b$

$\iff \dfrac{\ln a}{a} < \dfrac{\ln b}{b}$

$0 \leq \underbrace{a < b}_{} < e \implies \underbrace{\dfrac{\ln a}{a} < \dfrac{\ln b}{b}}_{}$

$\implies f(x) = \dfrac{\ln x}{x} \impliedby$

$$f'(x) = \frac{\cancel{1/x \cdot x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

$f'(x) = 0 \iff 1 - \ln x = 0 \implies \ln x = 1 \iff \boxed{x = e}$

| | 0 | | $e$ | |
|---|---|---|---|---|
| $f'(x)$ | | $+$ | | $-$ |
| $f(x)$ | $0$ | $\nearrow$ | | $\searrow$ |

$e^{-1}$ $\quad e^{2}$

$f$ es creciente en $(0, e) \implies$
$\forall a, b \in (0, e) \mid \boxed{a < b} \implies f(a) < f(b) \implies$
$\implies \boxed{\dfrac{\ln a}{a} < \dfrac{\ln b}{b}}$

b) $f$ es decreciente a $(e, +\infty) \implies$
$\forall a, b \in (e, +\infty) \implies \boxed{a < b} \implies f(a) > f(b)$

$\dfrac{\ln a}{a} > \dfrac{\ln b}{b} \iff$

$\boxed{a^b > b^a}$