2. Sean a y b números reales y sea A_n la matriz de $M_{n\times n}(\mathbb{R})$ definida como

$$A_n = \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ a & b & -1 & 0 & \dots & 0 \\ a^2 & ab & b & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a^{n-1} & a^{n-2}b & a^{n-3}b & a^{n-4}b & \dots & b \end{pmatrix} \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix}$$

A

(i)

para $n \in \mathbb{N}$ con $n > 2$. Determinar:

(a) El determinante de A_n.

(b) Las ecuaciones implícitas del subespacio vectorial de $\mathbb{R}^n$ generado por los vectores columna de A_n.

(c) La dimensión del espacio cociente $\mathbb{R}^n/Ker(f)$, con $f : \mathbb{R}^n \to \mathbb{R}^n$ la aplicación lineal con matriz asociada A_n.

(a)

$$|A_n| = \begin{vmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ a & b & -1 & 0 & \dots & 0 \\ a^2 & ab & b & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a^{n-1} & a^{n-2}b & a^{n-3}b & a^{n-4}b & \dots & b \end{vmatrix} \quad C_2 + C_1 \quad =$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ a & b+a & -1 & 0 & \dots & 0 \\ a^2 & a(b+a) & b & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a^{n-1} & a^{n-2}(a+b) & a^{n-3}b & a^{n-4}b & \dots & b \end{vmatrix} =$$

$$= (a+b) \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ a & 1 & -1 & 0 & \dots & 0 \\ a^2 & a & b & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a^{n-1} & a^{n-2} & a^{n-3}b & a^{n-4}b & \dots & b \end{vmatrix} =$$

$$= (a+b) \begin{vmatrix} 1 & -1 & 0 & \dots & 0 \\ a & b & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \end{vmatrix} =$$

$$= (a+b) \begin{vmatrix} a & b & & & \\ \text{—} & \text{—} & \text{—} & \text{—} & \text{—} \\ a^{n-2} & a^{n-3}b & a^{n-4}b & \text{—} & b \end{vmatrix} =$$

$$= (a+b)\,|A_{n-1}|$$

$$|A_n| = (a+b)\,|A_{n-1}|$$

$$|A_3| = \begin{vmatrix} 1 & -1 & 0 \\ a & b & -1 \\ a^2 & ab & b \end{vmatrix} = b^2 + a^2 + ab + ab =$$

$$= a^2 + 2ab + b^2 =$$

$$= (a+b)^2$$

$$|A_4| = (a+b)\,|A_3| = (a+b)^3$$

...

$$\boxed{|A_n| = (a+b)^{n-1} \qquad n > 2}$$

(b) C e.v. vectores columna de A_n

$$|A_n| = 0 \iff (a+b)^{n-1} = 0 \iff a = -b$$

$*$ Si $a \neq -b \Rightarrow |A_n| \neq 0 \Rightarrow n$ vectores son l.i.

$\Rightarrow \dim(\mathcal{C}) = n \Rightarrow \mathcal{C} = \mathbb{R}^n$.

(1)

$*$ Si ~~$a=-b$~~ $b = -a$

$$A_n = \left(\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & \cdots & 0 & x_1 \\ a & -a & -1 & 0 & \cdots & 0 & x_2 \\ a^2 & -a^2 & -a & -1 & \cdots & 0 & x_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ a^{n-1} & -a^{n-1} & -a^{n-2} & -a^{n-3} & \cdots & -a & x_n \end{array}\right) \begin{array}{l} \\ F_2 - aF_1 \\ F_3 - aF_2 \\ \\ \boxed{F_k - aF_{k-1}} \end{array}$$

$$\sim \left(\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & \cdots & 0 & x_1 \\ 0 & 0 & -1 & 0 & \cdots & 0 & x_2 - ax_1 \\ 0 & 0 & 0 & -1 & \cdots & 0 & x_3 - ax_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & 0 & \cdots & -1 & x_{n-1} - ax_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \boxed{x_n - ax_{n-1}} \end{array}\right)$$

$$\boxed{\mathcal{C} \equiv x_n - ax_{n-1} = 0}$$

(c)

* Si $a \neq -b \rightarrow |A_n| \neq 0 \rightarrow \ker(f) = \{0_n\}$

$$\dim(\ker(f)) = 0 \rightarrow$$

$$\boxed{\dim\left(\mathbb{R}^n / \ker f\right) = n}$$

* Si $a = -b$

$$\begin{array}{c}
\begin{array}{cccccc|c} x_1 & x_2 & x_3 & & \cdots & x_n & \end{array} \\
\left(\begin{array}{cccccc|c}
1 & -1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & -1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{array}\right)
\end{array}$$

$$x_3 = x_4 = x_5 = \cdots = x_n = 0$$

$$x_1 - x_2 = 0 \rightarrow \left\{\begin{array}{l} x_1 = x_2 \\ x_2 = x_2 \\ x_3 = 0 \\ x_4 = 0 \end{array}\right. \qquad x_2 \in \mathbb{R},$$

$$\dim(\ker f) = 1 \rightarrow \boxed{\dim\left(\mathbb{R}^n / \ker f\right) = n - 1}$$

$\dim(\ker f) = 1$

$$B_{\ker f} = \{(1,1,0,0,\dots 0)\}$$

$$B_{\mathbb{R}^n/\ker f} = \{e_i\}_{2\le i\le n}$$

$$e_i = (0,0,\dots,\overset{i}{1},\dots,0)$$